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A classification of subgroups of the Monster isomorphic to S_4 and an application

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Abstract

We classify all subgroups of the Monster isomorphic to S_4. We then use this classification to prove that there are no maximal subgroups of the Monster with socles isomorphic to $PSU_3(3)$, $PSL_3(3)$, $PSL_2(17)$, or $PSL_2(7)$.

Keywords: Sporadic simple groups; Monster group; Maximal subgroups

1. Introduction

The Monster $\mathbb{M}$ is the largest of the 26 sporadic simple groups. It has order

$$808017424794512875886459904961710757005754368000000000.$$

There are still many holes to be filled in our knowledge of its structure.

For example, the maximal subgroups are yet to be completely determined. There are eleven isomorphism types of simple (possible) subgroups of $\mathbb{M}$ which have not been fully classified. There are also many small local subgroups, e.g. S_4, that are not classified either.

In this paper we fill in a few of these gaps. We first classify all conjugacy classes of S_4's inside the Monster. The results on the S_4's then enable us to rule out various possible new maximal subgroups. The putative new maximals are those with socles isomorphic to $U_3(3)$, $L_3(3)$, $L_2(17)$, or $L_2(7)$.

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1.1. Outline of method

We begin by listing all conjugacy classes of A_4 in $\mathbb{M}$. It is likely that this is already well known, but the author was unable to find a convenient published classification of them.

We then look at outer-involution centralisers in groups isomorphic to $A_4.C_{\mathbb{M}}(A_4)$. Doing this for the different classes of A_4 gives possible S_4 centralisers in $\mathbb{M}$. Together with the $(2,3,4)$ structure constants, this is enough to classify S_4's for some class fusions.

The class fusions not dealt with by this method are the classes of S_4 with transpositions in Monster class $2B$ and with elements of order 3 in Monster classes $3B$ and $3C$. These are classified using the 196882 dimensional computer representation over $GF(3)$ of [2] as described in Section 4.4.

When we have complete lists of conjugacy classes of S_4's we can consider groups that are generated by S_4's. The groups $U_3(3)$, $L_3(3)$, $L_2(17)$, and $L_2(7)$ can all be generated by copies of S_4 intersecting in an S_3. In Section 5 we use this fact to classify the remaining conjugacy classes of these groups.

Section 6 gives the words for all elements used in Table 4.4.1 and Section 5. These can be used by the reader in conjunction with [2] to verify the results of the paper.

2. Background material

The computer construction used is described in [2]. Further techniques for using the construction were developed in [3] and [4]. We summarise them below.

2.1. The construction

The construction uses the 2-local subgroups and is over $GF(3)$. We first constructed two generators, c and d, for $G \cong 2_+^{1+24\cdot}\mathrm{Co}_1$, the centraliser of an element z in class $2B$. We then restricted this group to $K \cong 2^{2+11+22\cdot}\mathrm{M}_{24}$, the centraliser of a second involution. This was extended to $2^{2+11+22\cdot}(\mathrm{M}_{24} \times 3)$ by adjoining an extra generator, t, which cycles the three involutions in $Z(K)$.

We use five generators for K. These are a and b, which together generate $C_G(t) \cong 2^{11\cdot}\mathrm{M}_{24}$, and three involutions u, v, and w in $O_2(K)$.

Modulo the normal 2^{2+11}, the group K has quotient $\bar{K} \cong (2^{11} \times 2^{11}) : \mathrm{M}_{24}$. We defined three subgroups U, V, and W of K so that their images in this quotient group were the two direct factors and the diagonal subgroup of the $2^{11} \times 2^{11}$. The third generator t permutes the three subgroups U, V, and W, and so the involutions u, v, and w were chosen from the three subgroups so that they would also be permuted by t.

They were found in the centraliser of an element l of order 11.

The 196882 dimensional module for $2_+^{1+24\cdot}\mathrm{Co}_1$ has shape

$$\mathbf{298} \oplus \mathbf{98280} \oplus \mathbf{98304} \cong \mathbf{298} \oplus \mathbf{98280} \oplus (\mathbf{24} \otimes \mathbf{4096}).$$

The four modules **24**, **298**, **98280**, and **4096** are in fact modules for the double cover $2^{\cdot}G$. We store each element $g \in G$ as a file containing four matrices: $g_{\mathbf{24}}$, $g_{\mathbf{298}}$, $g_{\mathbf{98280}}$, and $g_{\mathbf{4096}}$, where $g_{\mathbf{n}}$ denotes "the image of some element g' in $\mathbf{n}$, where g' is a preimage of g in the double cover of $G = 2_+^{1+24\cdot}\mathrm{Co}_1$."

Table 1

Class fusion	$C_{\mathbb{M}}(2^2)$
AAA	$2^{2\cdot 2}\mathrm{E}_6(2)$
AAB	$2^{2+22\cdot}\mathrm{Co}_2$
ABB	$2^{1+16+1+8\cdot}\mathrm{O}_8^+(2)$
BBB	$2^{2+11+22\cdot}\mathrm{M}_{24}$
BBB	$2^{1+16+1+8+6}\mathrm{A}_8$
BBB	$2^{2+10+10}2\mathrm{M}_{12}$

The module for G has a different shape from that of the 4-group normaliser so t could not be stored in the same format as elements of G. Thus we store arbitrary elements as words in t and elements of G.

2.2. Techniques

Multiplication is performed by concatenating the words for the factors. Only elements of G can be multiplied together in a more useful way. Allowing an element to act on a vector is the only other permitted operation. In [3] we defined the *length* of a word as the number of occurrences of t in it. It takes approximately six seconds $\times$ word length to multiply a vector by a word using a Pentium II/450MHz processor.

Any word w in $\mathbb{M}$ which centralises z can be written in the same form at as our generators for G, i.e. as a file containing four matrices. But these elements are frequently found as much longer words. In [3] we gave a method for converting a word w of nonzero length in $C_{\mathbb{M}}(z)$ to the corresponding length zero word given in the more useful format. This trick is of fundamental importance in our work, as it is the only method we have for shortening words and hence preventing exponential growth of the words that we need.

Then in [4] we used this word-shortening trick to find a way of changing from one $2B$ centraliser (a post) to another. This can be done whenever one of the involutions is in the centraliser of another. We also showed how repeating this trick along a chain of involutions could in theory be used to shorten almost any word, although this is often too time consuming in practice.

It was shown in [4] that it is possible to change posts using a conjugating element of length at most 2. The first occurrence of t in this word conjugates the $2B$ involution into $O_2(G) \cong 2_+^{1+24}$. Then an element of G conjugates it to one of two known involutions in this group. Finally, either t or t^{-1} is used to conjugate it to z.

3. Theoretical S_4's

In this section, we classify as many classes of S_4 as we can without a computer. We do this by first classifying the A_4's. Then we use this classification to start the S_4's. The remaining cases are dealt with by computer in Section 4.

3.1. Conjugacy classes of A_4's

We begin by listing the Monster's conjugacy classes of 4-groups in Table 1. Then for each class of 2^2 we calculate the centraliser of each class of outer automorphism of order 3. This gives a list of all A_4 centralisers in $\mathbb{M}$. All classes of A_4 are listed in Table 2.

Table 2

Class fusion	$C_{\mathbb{M}}(\mathrm{A}_4)$	$C_{\mathbb{M}}(2^2)$
$2A, 3A$	$\mathrm{O}_{10}^{-}(2)$	$2^{2\cdot 2}\mathrm{E}_6(2)$
$2A, 3B$	$\mathrm{U}_5(2) \times \mathrm{S}_3$	$2^{2\cdot 2}\mathrm{E}_6(2)$
$2A, 3C$	${}^3\mathrm{D}_4(2) \cdot 3$	$2^{2\cdot 2}\mathrm{E}_6(2)$
$2B, 3A$	$2^{11\cdot}\mathrm{M}_{24}$	$2^{2+11+22\cdot}\mathrm{M}_{24}$
$2B, 3B$	$2^{\cdot}\mathrm{M}_{12}:2$	$2^{2+10+10}:2.\mathrm{M}_{12}.2$
$2B, 3B$	$2^{1+6}3^{1+2}4$	$2^{2+10+10}:2.\mathrm{M}_{12}.2$
$2B, 3B$	$2^{5+6\cdot}3^{\cdot}\mathrm{A}_6$	$2^{2+11+22\cdot}\mathrm{M}_{24}$
$2B, 3C$	$2^{1+4}(\mathrm{A}_4 \times \mathrm{A}_5)$	$2^{1+16+1+8+6}\mathrm{A}_8$
$2B, 3C$	$2^{1+8\cdot}\mathrm{A}_8$	$2^{1+16+1+8+6}\mathrm{A}_8$
$2B, 3C$	$2^{3+8}(3 \times \mathrm{L}_3(2))$	$2^{2+11+22\cdot}\mathrm{M}_{24}$
$2B, 3C$	$2^{8+2}3\mathrm{S}_3.2$	$2^{2+10+10}:2.\mathrm{M}_{12}.2$
$2B, 3C$	$2^{11}3\mathrm{S}_3$	$2^{1+16+1+8+6}\mathrm{A}_8$

Table 3

	$C_{\mathbb{M}}(\mathrm{A}_4)$	$C_{\mathbb{M}}(\mathrm{S}^4)$
$2A, 2A, 3A, 4B$	$\mathrm{O}_{10}^{-}(2)$	$\mathrm{S}_8(2)$
$2A, 2A, 3C, 4B$	${}^3\mathrm{D}_4(2).3$	${}^3\mathrm{D}_4(2).3$
$2A, 2B, 3A, 4A$	$2^{11\cdot}\mathrm{M}_{24}$	$2^{11\cdot}\mathrm{M}_{23}$
$2B, 2A, 3B, 4B$	$\mathrm{U}_5(2) \times \mathrm{S}_3$	$(\mathrm{A}_6 \times 6):2$
$2B, 2A, 3B, 4B$	$\mathrm{U}_5(2) \times \mathrm{S}_3$	$(\mathrm{A}_6 \times 2):2$
$2B, 2A, 3C, 4B$	${}^3\mathrm{D}_4(2).3$	$2^{1+4}3\mathrm{S}_3$

As all involutions in an A_4 are conjugate, we only need consider normalisers of the pure 4-groups. Counting eigenvalues of the diagonal elements of order 3 in the groups $\mathrm{M}_{24} \times 3$ and $2\mathrm{M}_{12} \times 3$ gives us each $C_{\mathbb{M}}(\mathrm{A}_4)$ in $2^{2+11+22\cdot}(\mathrm{M}_{24} \times 3)$ and $2^{2+10+10}(2\mathrm{M}_{12} \times 3)$. Then $(2,3,3)$ structure constants fill in the blanks and match centralisers with class fusions.

3.2. *Some* S_4*'s*

All S_4 centralisers can be found as centralisers of outer involutory automorphisms in groups $(\mathrm{A}_4.C_{\mathbb{M}}(\mathrm{A}_4))$. This method and the $(2,3,4)$ structure constants gives us all classes of S_4 containing $2A$'s, as shown in Table 3.

Note that the $(2A, 3, 4A/C/D)$ structure constants are zero in all cases except $(2A, 3A, 4A)$. The $2B$ pure cases are trickier so we leave them to be determined computationally in the next section.

4. Computational S_4's

In this section we use a computer to find representatives of all conjugacy classes of S_4 in $\mathbb{M}$ with transpositions in class $2B$ and elements of order 3 in classes $3B$ and $3C$.

We start by finding representatives of each type of S_3 and their centralisers. The S_4's are all obtained by adjoining extra elements to the S_3's and are identified by calculating their centralisers inside the centraliser of the corresponding S_3.

Table 4

	Class fusion	$C_{\mathbb{M}}(\mathrm{S}_4)$	R_1	BCA	$2^{1+4\cdot}\mathrm{A}_8$
R_2	BCB	$2^{1+3}3\mathrm{S}_3$	R_3	BCC	$2^{1+8+3}\mathrm{L}_3(2)$
R_4	BCC	$2^{2+3+3}\mathrm{L}_3(2)$	R_5	BCC	$2^{2+3+4}\mathrm{S}_3$
R_6	BCC	$2^{1+3\cdot}\mathrm{L}_3(2)$	R_7	BCC	$2^{2+4\cdot}\mathrm{S}_3$
R_8	BCC	$2^{1+3+2}:3$	R_9	BCD	2^{3+3}
R_{10}	BBA	$2\mathrm{M}_{12}$	R_{11}	BBB	$(\mathrm{A}_6 \times 6):2$
R_{12}	BBB	$(\mathrm{A}_6 \times 2):2$	R_{13}	BBC	$2^{2+4}\mathrm{S}_3$
R_{14}	BBC	$2^{2+4}\mathrm{S}_3$	R_{15}	BBC	$2 \times \mathrm{S}_4$
R_{16}	BBD	$2 \times \mathrm{S}_4$	R_{17}	BBD	$\mathrm{A}_5 \times 4$

Notation. We use several posts in this section. In each case we denote the post by P_i, the central involution by z_i and let k_i be the element conjugating z to z_i. The words for all elements used can be found in Section 6.

4.1. $\mathrm{S}_3 \times 2^{1+8\cdot}\mathrm{A}_9$

There is one class of S_3 in $\mathbb{M}$ containing $2B$ and $3C$ elements. It has centraliser $2^{1+8\cdot}\mathrm{A}_9$. Here we will find one of these S_3's. We will call it S.

We use the same $3C$ that we used in [4] given by the word g. Generators g_1 and g_2 for its centraliser $H = C_{\mathbb{M}}(S) \cong 2^{1+8}\mathrm{A}_9$ were found in [4]. Now we find one of the two involutions centralising $\langle g_1, g_2\rangle$ and inverting g. One of them is a $2A$ and one is a $2B$. It turns out that we have found the $2A$ so we multiply it by z to get the $2B$, because $\langle z\rangle = Z(H)$. This gives the element z_2. We have $S = \langle h, z_2\rangle$.

Later, we will examine S_4 centralisers inside the z_2 post, P_2, so we find the post changer k_2. We shorten the words $h_1^{k_2}$ and $h_2^{k_2}$ to see H as a subgroup of the new post.

4.2. Finding S_4*'s in posts*

All involutions in $C_{\mathbb{M}}(S)$ are in class $2B$, so representatives of all classes of S_4 that contain S and have even-order centralisers can be found in posts.

Structure constants in the character table of G (available in GAP [8]) show that we cannot extend S to all of the S_4 classes solely by working in G. The full $(2B, 3C, 4A)$ and $(2B, 3C, 4C)$ structure constants are $1/645120$ and $839/98304$, respectively, but S_4's in G only account for some of them. There are no copies of S_4 in G containing S and elements of class $4D$.

There are three classes of involutions in $2^{1+8\cdot}\mathrm{A}_9$. Each gives rise to a different Monster class of groups isomorphic to $2 \times \mathrm{S}_3$. We find involutions z_3 and z_4 extending S to representatives of both the other classes of $2 \times \mathrm{S}_3$ and conjugate S into both the new posts.

We look for the S_4's by considering involutions in $C_P(z_2)$, where P is one of the posts P_1, P_3, or P_4. Whenever an involution r was found extending S to a representative R of a new post-class of S_4, we conjugated it into P_2 and calculated $C_H(v) = C_{\mathbb{M}}(R)$. Computations with MAGMA [1] in **196560P**, a 196560 point permutation representation of $P_2/Z(P_2)$, gave the structures of these centralisers.

The classes $R_1, \ldots, R_5$ and $R_7, \ldots, R_9$ of S_4 were found in this way. We denote the involution found extending S to R_i by r_i. The classes are listed in Table 4.

4.3. An S_4 *in a Held group*

In the previous subsection we accounted for the whole structure constant for class fusions $(2B, 3C, 4*)$, where $*$ is one of A, B, and D. A portion of the $(2B, 3C, 4C)$ structure constant remains. In this section we deal with these final cases.

The remaining structure constant is $1/2688$. This suggests that at least one of the classes yet to be found has centraliser order divisible by 7. If this is the case, looking at powers of elements of order 28 tell us that any 7 centralising an S_4 of type $(2B, 3C, 4C)$ must be in class $7A$. So we will look for an S_4 inside a copy of He.

We begin by finding a suitable 7. There is only one 7-class in H so we can use any element of this class. We choose the element δ_4. Copies of $\mathrm{S}_4 \leqslant C_{\mathbb{M}}(s)$ with $S \leqslant R$ intersect $C_{P_2}(\delta_4)$ in a 4-group, so we proceed by looking at involutions in $C_{P_2}(\delta_4) \cong 2^{1+6} {}^{\cdot}\mathrm{L}_3(2)$.

There are eight classes of involution in $2^{1+6}.\mathrm{L}_3(2)$. We disregard the central class as this consists of z_2. We can also ignore one class out of each pair of classes that are equal modulo the centre, and those that consist of involutions generating a non $2B$-pure 4-group with z_2.

We found generators δ_{12} and δ_{13} for $C_{P_2}(\delta_4)$, then computed $|C_H(r)|$ for each involution r in the classes not yet eliminated. The involutions for which this order was at least 2688 were then tested to find which extended S to S_4's.

We found the class R_6. This completes the classification of all S_4's containing S.

4.4. $(2B, 3B)$ S_4*'s*

The $2B, 3B$ case is very similar to the previous one, so we describe it here in less detail.

Again, there is only one class of S_3 with the required class fusions. The centraliser is $3^6 : 2\mathrm{M}_{12}$.

We find an S_3 called T with generators h and z_6. This time the whole centraliser is not in the same post as T. We find two generators h_1 and h_2 in G and a third in a new post, P_5.

The posts G, P_5, and P_7 were searched in an attempt to find S_4's. We used the method of the previous subsection, namely adjoining involutions in $C_P(z_6)$ for each post P, and we also looked at different elements of order 3 inverted by z_6. This two-pronged attack yielded all classes of S_4 containing T.

The two classes of BBC S_4's with isomorphic centralisers can distinguished by the centralisers in $\mathbb{M}$ of the transposition centralisers in each R_i, $i \in \{13, 14\}$. We have $C_{\mathbb{M}}(C_{R_{13}}(z_6)) \cong 2^{2+10+10} : 2\mathrm{M}_{12}.2$ and $C_{\mathbb{M}}(C_{R_{13}}(z_6)) \cong 2^{1+16+1+8+6}\mathrm{A}_8$.

5. The application

The groups $\mathrm{L}_2(7)$, $\mathrm{L}_2(17)$, $\mathrm{L}_3(3)$, and $\mathrm{U}_3(3)$ can be generated by conjugate S_4's intersecting in an S_3. These are some of the subgroups of $\mathbb{M}$ which are still not fully classified. In particular, it is possible that their normalisers could be new maximal subgroups of $\mathbb{M}$.

5.1. Method

Table 3 in [6] gives class fusions for which we have no classification of subgroups of $\mathbb{M}$ isomorphic to these groups. From this list we see that the generating S_4's intersect in an S_3 with class fusion $2B, 3C$ for $\mathrm{L}_2(7)$ and $\mathrm{U}_3(3)$, $2B, 3B$ for $\mathrm{L}_2(17)$ and either $2B, 3B$ or $2B, 3C$ for $\mathrm{L}_3(3)$.

The above shows that representatives of all unclassified conjugacy classes of $L_2(7)$, $L_2(17)$, $L_3(3)$, and $U_3(3)$ can be generated by one of the copies of S_4 found computationally in the previous section, together with at least one of its conjugates under the centraliser of its S_3.

First we consider the cases that can be generated by a pair of S_4's. These are $L_2(7)$, $L_2(17)$, and $L_3(3)$. These results can then be used for $U_3(3)$, as this group requires a minimum of three S_4's, with each pair generating an $L_2(7)$.

5.2. *Double cosets*

Testing all conjugates of S_4 by the centraliser of an S_3 is impossible, as the centraliser orders for the $3B$ and $3C$ cases are 92897280 and 138568320, respectively.

In [5], we looked at groups generated by conjugate A_5's intersecting in a D_{10}. We showed that it was sufficient to conjugate the A_5 by one representative from each double coset $C_{\mathbb{M}}(A_5)C_{\mathbb{M}}(D_{10})C_{\mathbb{M}}(A_5)$. We can apply the same argument here.

We shall find double coset representatives of $C_{\mathbb{M}}(S_4)$ in $C_{\mathbb{M}}(S_3)$ for each class of S_4. The double coset representatives were easily found in [5] as the D_{10} centraliser is small. In our case, a naive approach cannot be used as we are now working on a larger scale.

For any groups G and H with $H \leqslant G$, the double cosets of H in G correspond to the orbits of H on the cosets of H in G. In practical terms, this means that we can find double coset representatives by taking the permutation representation of G on the cosets of H, calculating the orbits of H and choosing one of the fixed points to be point 1, then finding elements of G mapping point 1 into each of the other H-orbits. These elements are our double coset representatives.

The calculation was performed in MAGMA using **196560P** restricted to $C_{\mathbb{M}}(S_3)$. In most cases, we chose a suitable $C_{\mathbb{M}}(S_4)$-orbit of points λ, calculated the orbit of orbits $\Lambda = \lambda^{C_{\mathbb{M}}(S_3)}$ then stored only the orbits of $C_{\mathbb{M}}(S_4)$ on Λ.

Even this required more space than poor MAGMA was able to cope with when the S_4 was R_9, R_{15}, or R_{16}. In these cases, we enlarged the group $C_{\mathbb{M}}(S_4)$ by adjoining a normal subgroup N_i $i \in \{9, 15, 16\}$. This reduces the number of double cosets sufficiently, but must be compensated for as described in the following subsection.

5.3. *Testing cases*

Here are presentations for $L_2(7)$, $L_3(3)$, $L_2(17)$, and $U_3(3)$:

$$L_2(7) \cong \big\langle b, c, d \;\big|\; b^2, c^2, d^3, \big((bc)^2 d\big)^2, (bd)^3, (cd)^3, \big((bc)^2 b\big)^2, \big((bc)^2 c\big)^2, (bc)^4, (bcd)^7, (cbd)^7, \big((bc)^2 bd\big)^4, \big((bc)^2 cd\big)^4\big\rangle.$$

We have $\langle b, d\rangle \cong \langle c, d\rangle \cong S_4$ and $\langle b, c\rangle \cong D_8$.

$$L_3(3) \cong \big\langle b, c, d \;\big|\; b^2, c^2, d^3, \big((bc)^3 d\big)^2, (bd)^3, (cd)^3, \big((bc)^3 b\big)^2, \big((bc)^3 c\big)^2, (bc)^6, (bcd)^8, \big((bc)^3 bd\big)^4, \big((bc)^3 cd\big)^4\big\rangle.$$

Again we have $\langle b, d\rangle \cong \langle c, d\rangle \cong S_4$ but this time $\langle b, c\rangle \cong D_{12}$.

$$\mathrm{L}_2(17) \cong \big\langle b, c, d \;\big|\; b^2, c^2, d^3, (bc)^8, (bcd)^4, (cbd)^4, (bd)^3, (cd)^3, \big((bc)^4d\big)^2, \big((bc)^4b\big)^2, \big((bc)^4c\big)^2\big\rangle.$$

The same idea, with $\langle b, c\rangle \cong \mathrm{D}_{16}$.

$$\mathrm{U}_3(3) \cong \big\langle a, b, c, d \;\big|\; a^2, b^2, c^2, d^3, \big((bc)^2d\big)^2, (bd)^3, \big((ac)^2d\big)^2, (ad)^3, (cd)^3, \big((bc)^2b\big)^2, \big((bc)^2c\big)^2, \big((ac)^2a\big)^2, \big((ac)^2c\big)^2, (ac)^4, (bc)^4, (acd)^7, (bcd)^7, (cbd)^7, (cad)^7, \big((bc)^2bd\big)^4, \big((bc)^2cd\big)^4, \big((ac)^2ad\big)^4, \big((ac)^2cd\big)^4, (abc)^4, (abc)^i2(bc)^2, (abcd)^12\big\rangle.$$

Together a, c, d and b, c, d satisfy the relations for $\mathrm{L}_2(7)$ given above. We add on the three relations $(abc)^4$, $(abc)^2(bc)^2$ and $(abcd)^{12}$ to get $\mathrm{U}_3(3)$.

In each of the four presentations, the relations involving only b and d are satisfied by the pairs (r_i, g) for $1 \leqslant i \leqslant 9$ and (r_i, h) for $10 \leqslant i \leqslant 17$. Also, g (or h) together with the conjugates of the appropriate r_i under $C_{\mathbb{M}}(S)$ ($C_{\mathbb{M}}(T)$) satisfy the relations involving only c and d, or a and d. Calculations inside $\mathrm{L}_2(7)$, $\mathrm{L}_2(3)$, $\mathrm{L}_2(17)$, and $\mathrm{U}_3(3)$ show that the remaining relations are satisfied by all such sets if and only if they generate the whole group.

For $1 \leqslant i \leqslant 17$, $i \notin \{9, 15, 16\}$, we look at the conjugates of each r_i under the double coset representatives found in the previous subsection. In the cases $i \in \{9, 15, 16\}$ each double coset representative $\mu \in (N_iC_{\mathbb{M}}(R_i))C_{\mathbb{M}}(\mathrm{S}_3)C_{\mathbb{M}}(R_i)N_i$ is a disjoint union of k double cosets $\mu_j \in C_{\mathbb{M}}(R_i)C_{\mathbb{M}}(\mathrm{S}_3)C_{\mathbb{M}}(R_i)$, $1 \leqslant j \leqslant k$ so pre- and post-multiplying each μ by all elements of N_i ensures that we have all μ_j. So in these cases we look at each $r_i^{\mu_j}$.

In each case we test whether we can substitute the new element for c in any of the above presentations. The first tests are the relations $(bc)^n = (bc)^{n/2}b = (bc)^{n/2}c$. These can be done inside the posts P_2 and P_6 and are therefore very cheap. Most possibilities can be rejected at this stage. This leaves a relatively trivial number of cases to be tested slowly using our Monster matrices.

5.4. Results

The classes of subgroups isomorphic to $\mathrm{L}_2(7)$, $\mathrm{L}_3(3)$, $\mathrm{L}_2(17)$, and $\mathrm{U}_3(3)$ found are shown in Table 5. The first column gives the isomorphism type of group found and the second the structure of the centraliser in $\mathbb{M}$. The column headed 'Maximal' lists a maximal subgroup containing the normaliser of the groups and the final column gives a generating set for a representative of the conjugacy class.

Note that the normalisers of all groups in the table are contained in known maximal subgroups.

6. Words

We feel that it is important to provide the words for all elements used in the proof of our results. However, they can make the text unreadable if inserted throughout. So all words used are given in this section.

The words ϕ_i are from the MeatAxe [7] functions "fro" and "fro2." They are also printed in full in [4] and [3]. To save space, we denote the product $\phi_i(x, y)^m\phi_j(x, y)^n$ by $\phi_{i^m, j^n}(x, y)$.

Table 5

G	$C_{\mathbb{M}}(G)$	Maximal	Generators
$L_2(7)$	$2^{\cdot}A_7$	$2_+^{1+24\cdot}Co_1$	R_1, x_1
$L_2(7)$	$Q^{8\cdot}S_3$	$2_+^{1+24\cdot}Co_1$	R_7, x_2
$L_2(7)$	3	$3Fi_{24}' : 2$	R_8, x_3
$L_2(7)$	Q_8	$2_+^{1+24\cdot}Co_1$	R_9, x_4
$L_3(3)$	2	$2_+^{1+24\cdot}Co_1$	R_9, x_5
$L_3(3)$	S_3	2B	R_{16}, x_9
$L_3(3)$	$3^2 : 2S_4$	$2_+^{1+24\cdot}Co_1$	R_{10}, x_{10}
$U_3(3)$	$(2 \times 3^2)^{\cdot}4$	$2_+^{1+24\cdot}Co_1$	R_1, x_1, x_6
$U_3(3)$	$2^{\cdot}A_5$	$2_+^{1+24\cdot}Co_1$	R_1, x_1, x_7
$U_3(3)$	3	$3^{\cdot}Fi_{24}' : 2$	R_8, x_3, x_8
$L_2(17)$	2	2B	x_{11}

6.1. Words from Sections 4.1–4.3

$g = \phi_{22}(a,b)^7$

$m = (c)^{20}$

$g_1'' = (\phi_{1,23,16}(m,g)tm^2)^{10}$

$g_{2''} = (\phi_{9,7}(m^d, g)(m^d)^4)^2$

$g_1' = g_1'' g g^{g_1''}$

$g_2' = g_2'' g g^{g_2''}$

$g_2 = g_1' g_2' g_1'$

$y_1 = \phi_{20}(g_1, g_2)^3$

$y_2 = (\phi_4(g_1, g_2) y_1^{-1} y_1^{\phi_4(g_1,g_2)})^2$

$y_3 = (\phi_7(c,d) y_1^{-1} y_1^{\phi_7(c,d)})^{10}$

$y_4 = (\phi_5(c,d) y_1^{-1} y_1^{\phi_5(c,d)})^{10}$

$y_5 = y_3 (y_2^{-1} y_2^{y_3})^7$

$y_6 = \phi_4(y_4, y_5)(y_2^{-1} y_2^{\phi_4(y_4,y_5)})^3$

$z_2 = z\phi_{3,1,11,1}(g_1, g_2)^3 \phi_{7,20,2,2}(y_5, y_6)$

$k_2 = c(u^{(ab)^3} u z_2^c)^2 t \phi_{5^5,3^3,7^9,1^{-1},3^{-26},14^{-34}}(c,d) t^{-1}$

$z_3 = g_1^6$

$k_3 = \phi_{13^3,19,23^7,1^{-1},3^{-8},13^{-16}}(c,d)t$

$z_4 = \phi_{10}(g_1, g_2)^3$

$k_4 = (uz_4)^4 t \phi_{17^4,16^2,20^8,1^{-1},3^{-26},7^{-10}}(c,d) t^{-1}$

$\delta_1 = \phi_6(c,d)[z_2^{k_4}, \phi_6(c,d)]^{17}$

$\delta_2 = \phi_{16}(c,d)[z_2^{k_4}, \phi_{16}(c,d)]^{16}$

$\delta_3 = ((g_1, g_2)^{k_2})^{15}$

$\delta_4 = \phi_9(g_1^{k_2}, g_2^{k_2})^2$

$\delta_5 = c^{\phi_{9,2}(c,d)}$

$\delta_6 = \delta_5^{\phi_{7,1}(\delta_5,\delta_4)}$

$\delta_7' = \phi_{15,16,14,18}(\delta_6, \delta_4)$

$\delta_8' = \phi_{17,19^2,1}(\delta_6, \delta_4)$

$\delta_9' = c^{\phi_{15,10}(c,d)}$

$\delta_{10}' = \delta_9^{\phi_{7,3}(\delta_9,\delta_4)}$

$\delta_{11}' = \phi_{12,4,7,20^2,11,16,17}(\delta_{10}, \delta_4)$

$\delta_7' = \delta_4^5 \delta_7'(\delta_4, \delta_4^{\delta_7'})^3$

$\delta_8' = \delta_4^5 \delta_8'(\delta_4, \delta_4^{\delta_8'})^3$

$\delta_{11}' = \delta_4^5 \delta_{11}'(\delta_4, \delta_4^{\delta_{11}'})^3$

$\delta_{12} = \delta_7 \delta_{11}$

$\delta_{13} = \delta_3 \delta_8$

$\delta_{14} = (\delta_7 \delta_{11} \delta_3 \delta_{11})^2$

$\gamma_1 = ((z_2^{k_3} z_2^{k_3 \phi_8(c,d)})^3)^{k_3^{-1}}$

$\gamma_2 = ((z_2^{k_3} z_2^{k_3 \phi_{3,1}(c,d)})^3)^{k_3^{-1}}$

$\gamma_3 = ((z_2^{k_3} z_2^{k_3 \phi_{9,3}(c,d)})^3)^{k_3^{-1}}$

$\gamma_4 = ((z_2^{k_3} z_2^{k_3 \phi_{22,2}(c,d)})^3)^{k_3^{-1}}$

$\gamma_5 = ((z_2^{k_3} z_2^{k_3 \phi_{14,2}(c,d)})^3)^{k_3^{-1}}$

$\gamma_6 = ((z_2^{k_3} z_2^{k_3 \phi_{15,4}(c,d)})^3)^{k_3^{-1}}$

$\gamma_7 = ((z_2^{k_3} z_2^{k_3 \phi_{16,4}(c,d)})^3)^{k_3^{-1}}$

$r_1 = (z_2(\gamma_{5,1,2,1,5,2} g)^2)^2$

$r_2 = (z_2(\gamma_{5,2,1}^2 g)^2)^2$ $\quad r_3 = (z_2(\gamma_{5,2,5,1,2,1} g)^2)^2$
$r_4 = (z_2(\gamma_{2,5}^2 \gamma_{2,1}^2 g)^2)^2$ $\quad r_5 = (z_2(\gamma_{1,4,1,3,1,3,4,1} g)^2)^2$
$r_6 = \delta_{14}^{\phi_{11,4}(\delta_{12},\delta_{13})k_2^{-1}}$
$r_7 = \phi_{9,4}(z_2^{k_4}, \phi_{23,16,20,10,13,1,13,17,5,5,2,18,3,7,13}(\delta_1,\delta_2) g^{k_4} z_2^{k_4})$
$r_8 = (\delta_1^{12})^{\phi_{12,12,6,19,19,8,14,14}(\delta_1,\delta_2)k_2^{-1}}$ $\quad r_9 = (z_2(\gamma_{4,6,4,3,5,4,6,4,1,3} g)^2)^2$

6.2. Words from Section 4.4

$h = d^4$ $\quad \beta_3 = \phi_{5,3,7,4,8,4}(c,d)^{10}$
$\beta_1' = \phi_{15,1,23,6,23,6}(c,d)^{30}\beta_3$ $\quad \beta_2' = \phi_{9,8,23,8,9,13}(c,d)^{14}$
$\beta_1 = \beta_1' d^2 (d^2)^{\beta_1'}$ $\quad \beta_2 = \beta_2' d^2 (d^2)^{\beta_2'}$
$z_6'' = \phi_{18,4,22,18,9,14}(c,d)^6$ $\quad z_6' = z_6'' d^2 (d^{-2})^{z_6''}$
$z_6 = z_6' \beta_2 \beta_3 d^2 (d^2)^{\beta_3}$
$k_6 = cd(z_6^{cd} u)^5 t \phi_{13^6,5^3,11^2,1^{-1},3^{-31},13^{-16}}(c,d) t^{-1}$
$h_1 = \beta_1 z_6 z_6^{\beta_1}$
$h_2 = \phi_{20}(\beta_1,\beta_2)(z_6 z_6^{\phi_{20}(\beta_1,\beta_2)})^4 \phi_{23}(\beta_1,\beta_2)(z_6 z_6^{\phi_{23}(\beta_1,\beta_2)})^4$
$z_5 = z h_1^4$
$k_5 = c(z_5^c u^{(ab)^3} u)^2 t \phi_{6^6,15^8,20^8,1^{-1},2^{-2},22^{-7}}(c,d) t^{-1}$
$\beta_4' = \phi_{11,14,20,14,5,13,10,2}(c,d)^{10}$ $\quad \beta_4 = \beta_4' h^{k_5} h^{k_5 \beta_4'}$
$h_3 = ((z_6^{k_5} z_6^{k_5\beta_4})^3)^{k_5^{-1}}$ $\quad \beta_5 = cd(z_6^{k_5} z_6^{k_5 cd})^{17}$
$\beta_6 = \phi_7(c,d)(z_6^{k_5} z_6^{k_5\phi_7(c,d)})^{16}$ $\quad \beta_7 = cd(z_6 z_6^{cd})^5$
$\beta_8 = \phi_9(c,d)(z_6 z_6^{\phi_9(c,d)})^{16}$ $\quad \beta_{10} = (z_6^{k_5} z_6^{k_5\phi_{15}(c,d)})^4$
$\beta_{11} = (z_6 z_6^{k_5\phi_7(c,d)})^{11}$ $\quad \beta_9 = (z_6 z_6^d)^2$
$r_{10} = (z\beta_5^{\phi_{10,17,6,5,12,13}(\beta_5,\beta_6)})^{k_5^{-1}}$ $\quad r_{13} = (((\beta_{11}^{\phi_{3,6,2,2,13,9}(\beta_5,\beta_6)})^2 z_6^{k_5} h^{k_5})^2)^{k_5^{-1}}$
$r_{11} = (\phi_{3,13}(\beta_7,\beta_8)^6)^{\phi_{5,11,1,15,17,18}(\beta_7,\beta_8)}$ $\quad r_{12} = (\beta_7^6)^{\phi_{15,1,16,3,9,18}(\beta_7,\beta_8)}$
$r_{17} = (((\beta_{10}^{\phi_{11,5,10,15,10,3}(\beta_5,\beta_6)} z_6^{k_5})^2 z_6^{k_5})^2)^{k_5^{-1}}$ $\quad r_{16} = (\beta_9^{\phi_{3,7,16,7,8,1}(\beta_7,\beta_8)} z_6 h)^2$
$r_{14} = ((\beta_9^{\phi_{19,5,10,17,4,21,6,8}(\beta_7,\beta_8)} z_6)^2 z_6 h)^2$ $\quad r_{15} = ((h^{\phi_{15,6,22,11,8,6,15,16}(\beta_7,\beta_8)} z_6)^2 z_6 h)^2$

6.3. Words from Section 5

$x_1 = r_1^{\phi_{13,10,2,13,14,23}(g_1,g_2)}$ $\quad x_2 = r_7^{\phi_{9,17,4,23,15,15}(g_1,g_2)}$
$x_3 = r_8^{\phi_{23,3,10,20,13,23}(g_1,g_2)}$ $\quad x_4 = r_9^{\phi_{3,23,6,10,22,14}(g_1,g_2)}$
$x_5 = r_9^{\phi_{16,7,17,5,18,2}(g_1,g_2)}$ $\quad x_6 = r_1^{\phi_{17,16,3,12,10,15}(g_1,g_2)}$
$x_7 = x_1^{\phi_{12,4,9,21,5,7}(g_1,g_2)}$ $\quad x_8 = x_3^{\phi_{1,16,18,7,4,23}(g_1,g_2)}$
$\eta_1 = h_2^4 (h_2^{-4})^{h_1}$ $\quad x_9 = r_{16}^{\eta_1 \phi_{21,7,12,7,1,3}(h_1,h_2h_3)\eta_1^2}$
$x_{10} = r_{10}^{\phi_{1,5,22,13,18,18}(h_1,h_2h_3)}$ $\quad \eta_2 = h_2^4$
$\eta_3 = h_2^{h_1}$ $\quad x_{11} = r_{15}^{\eta_3 \phi_{3,12,10,21,5,15}(h_1,h_2h_3)\eta_3^2}$

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